Assignment 2

CMPS 392

Useful Formulas

Given: the sigmoid function:

$$\sigma(x) = \frac{1}{1 + exp(-x)}$$

and the softplus function

$$\zeta(x) = log(1 + exp(x))$$

Softplus means it is a soft version of $x^+ = max(0, x)$

Plot the curves of these functions and prove the following properties:

$$\sigma(x) = \frac{\exp(x)}{\exp(x) + \exp(0)}$$

$$\frac{d}{dx}\sigma(x) = \sigma(x)(1 - \sigma(x))$$

$$1 - \sigma(x) = \sigma(-x)$$

$$\log \sigma(x) = -\zeta(-x)$$

$$\forall x \in (0, 1),\ \sigma^{-1}(x) = \log(\frac{x}{1 - x})$$

$$\forall x > 0,\ \zeta^{-1}(x) = \log(\exp(x) - 1)$$

$$\zeta(x) = \int_{-\infty}^{x} \sigma(y)dy$$

$$\zeta(x) - \zeta(-x) = x$$

Expectation and Variance

Prove that the following are valid probability mass functions and compute their expectation and variance:

- $P(x) = \begin{cases} \phi & \text{if } x = 1 \\ 1 - \phi & \text{if } x = 0 \end{cases} \quad \phi \in [0, 1]$
- $P(k) = \binom{n}{k} \phi^k (1 - \phi)^{n-k} \quad \text{for} \quad 0 \leq k \leq n \quad \phi \in [0, 1]$
- $P(k) = \phi(1 - \phi)^{k-1} \quad \text{for} \quad k = 1, 2, \ldots \quad \phi \in [0, 1]$
- $P(k) = e^{-\lambda} \frac{\lambda^k}{k!} \quad \text{for} \quad k = 1, 2, \ldots \quad \lambda > 0$

Prove that the following are valid probability density functions and compute their expectation and variance:

- $p(x) = \frac{1}{b - a} \quad \forall x \in (a, b)$
- $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
- $p(x) = \lambda e^{-\lambda x} \quad x \geq 0, \quad \lambda > 0$

Spam detection

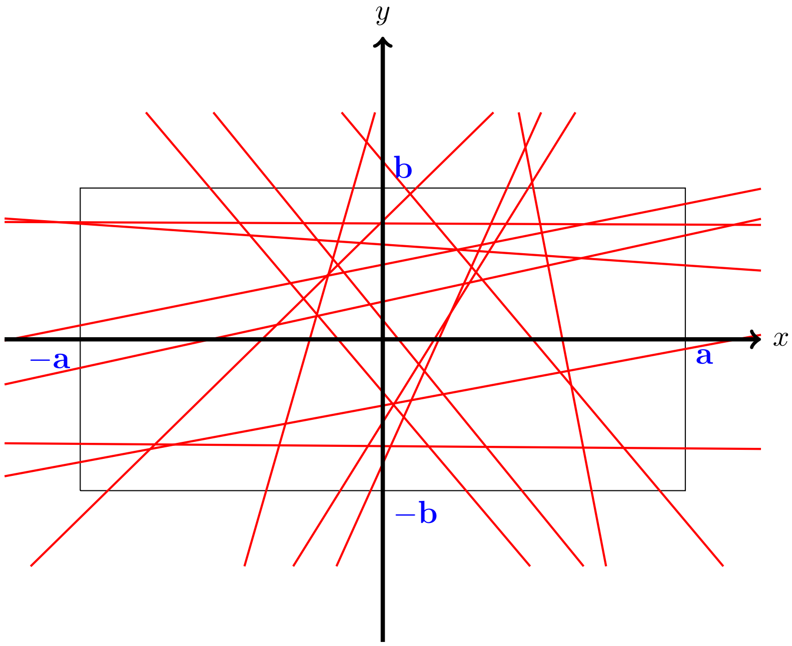
Suppose you receive an e-mail message with the subject "PRIZE AWARD". You have been keeping statistics on your e-mail, and have found that while only 10% of the total e-mail messages you receive are spam, 50% of the spam messages have the subject "PRIZE AWARD" and 2% of the non-spam messages have the subject "PRIZE AWARD". What is the probability that the message is spam?

Birthday Paradox

Imagine a society where families want to have the least amount of kids as long as they have a boy. What is the boys to girls ratio in such society?

Programming Assignment

We want to draw lines in a rectangular area in 2D in the most uniform way.



Consider the following methods:

- Sample $x_1, x_2 \sim \mathcal{U}[-a, a], \quad y_1, y_2 \sim \mathcal{U}[-b, b]$, and draw the line connecting point (x_1, y_1) and (x_2, y_2).
- Sample $r \sim \mathcal{U}[0, \sqrt{a^2 + b^2}]$ and $\theta \sim \mathcal{U}[0, 2\pi]$, and draw the line at angle θ and distance r from the origin.
- Sample $x \sim \mathcal{U}[-a, a],\ y \sim \mathcal{U}[-b, b],\ \gamma \sim \mathcal{U}[0, 2\pi]$, and draw the line passing through point (x, y) and having slope γ.

Apply for $a = 2,\ b = 1$. Draw around 100 lines for each method and compare. Which one looks the most uniform to you? If none of the above methods is satisfying, you can propose your own method.